

Cosmic ray transport theory in partially turbulent space plasmas with compressible magnetic turbulence

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ABSTRACT

Recently a new transport theory of cosmic rays in magnetized space plasmas extending the quasilinear approximation to the particle orbit has been developed for the case of an axisymmetric incompressible magnetic turbulence. Here we generalize the approach to the important physical case of a compressible plasma. As previously obtained in the case of an incompressible plasma we allow arbitrary gyrophase deviations from the unperturbed spiral orbits in the uniform magnetic field. For the case of quasi-stationary and spatially homogeneous magnetic turbulence we derive in the small Larmor radius approximation gyro-phase averaged cosmic ray Fokker-Planck coefficients. Upper limits for the perpendicular and pitch-angle Fokker-Planck coefficients and for the perpendicular and parallel spatial diffusion coefficients are presented.

Subject headings: cosmic rays – diffusion – magnetic fields – plasmas – turbulence

1. Introduction

The study of the cosmic ray transport in turbulent magnetic fields is crucial in many aspects of high energy astrophysics, such as the efficiency of cosmic ray diffusive shock acceleration, the modulation and penetration of low energy cosmic rays in the heliosphere and their confinement and escape from the Galaxy.

A new theory of cosmic ray transport in magnetized plasmas extending the quasilinear approximation to the particle orbit has been recently published by one of us (Schlickeiser, 2011) (hereafter Paper 1). In Paper 1 the transport parameters of energetic charged particles in turbulent magnetized cosmic plasmas were derived for the case of an incompressible plasma, i.e. plasmas for which the component of the magnetic turbulence, $\delta B_z = 0$, parallel to the guide magnetic field, $\vec{B}_0 = B_0 \vec{e}_z$, is set to zero. Here we present the generalization of the theory to the case of compressible magnetic turbulence with $\delta B_z \neq 0$.

In Section 2 we briefly review the theory developed in Paper 1. In Section 3 we obtain the gyro-phase averaged cosmic ray Fokker-Planck coefficients for a quasi-stationary, spatially homogenous turbulence under a Corrsin-type assumption on the nature of generalized orbits (Corrsin 1959, Salu and Montgomery, McComb 1990). Simplified formulas for the gyro-phase averaged cosmic ray Fokker-Planck coefficients are obtained in Section 4 assuming that the magnetic turbulence is asymmetric, while the quasilinear limit of the coefficients is shown in the Appendix. In Section 5 from the Fokker-Planck coefficients we derive upper and lower limits for the perpendicular and parallel spatial diffusion coefficients. In Section 6 we compare the relative importance of mirror forces and turbulent scattering for the cosmic ray transport in interstellar plasmas.

2. The gyro-averaged Fokker-Planck equations

2.1. Equations of motion of a particle in magnetic fields

For the following treatment we remind shortly the equation of motion of charged particles of mass m , charge q , and Lorentz factor $\gamma = \sqrt{1 + (p/mc)^2}$ in a uniform guide magnetic field $\vec{B}_0 = B_0 \vec{e}_z = (0, 0, B_0)$. A random magnetic field, $\delta \vec{B}$, is superposed to the guide field.

$$\dot{\vec{p}} = \frac{q}{m\gamma c} \vec{p} \times [\vec{B}_0 + \delta \vec{B}], \quad \dot{\vec{x}} = \vec{v} = \frac{\vec{p}}{\gamma m} \quad (1)$$

The scalar product of Eq. (1) with \vec{p} readily yields $p = |\vec{p}| = \text{const.}$, $v = \text{const.}$ and $\gamma = \text{const.}$. Introducing the constant relativistic gyrofrequency $\Omega = \frac{qB_0}{\gamma mc}$ and scaling the turbulent fields in units of B_0 , $\delta \vec{b} = \frac{\delta \vec{B}}{B_0}$ we obtain

$$\frac{d}{dt} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \Omega \begin{pmatrix} v_y(1 + \delta b_z) - v_z \delta b_y \\ -v_x(1 + \delta b_z) + v_z \delta b_x \\ v_x \delta b_y - v_y \delta b_x \end{pmatrix} \quad (2)$$

For the time evolution of the particle pitch-angle cosine $\mu = v_z/v$ and phase $\phi = \arctan(v_y/v_x)$ this implies

$$\frac{d\mu}{dt} = h_\mu(t) = \frac{\Omega}{v} (v_x \delta b_y - v_y \delta b_x) = \Omega \sqrt{1 - \mu^2} (\cos \phi \delta b_y - \sin \phi \delta b_x), \quad (3)$$

and

$$\frac{d\phi}{dt} = -\Omega + h_\phi(t), \quad h_\phi(t) = -\Omega \delta b_z + \frac{\Omega \mu}{\sqrt{1 - \mu^2}} (\cos \phi \delta b_x + \sin \phi \delta b_y). \quad (4)$$

with the two random forces $h_\mu(t)$ and $h_\phi(t)$.

In the coordinates of the guiding center

$$\vec{X} = (X, Y, Z) = \vec{x} + \frac{\vec{v} \times \vec{e}_z}{\Omega} = \vec{x} + \frac{1}{\Omega} \begin{pmatrix} v_y \\ -v_x \\ 0 \end{pmatrix} \quad (5)$$

Eqs. (2) become

$$\frac{d}{dt} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} v_z \delta b_x - v_x \delta b_z \\ v_z \delta b_y - v_y \delta b_z \\ v_z \end{pmatrix}. \quad (6)$$

Indicating $X_i = [X, Y]$ with $i, j = 1, 2$, Eq.(6) provide the two additional random force terms $h_i(t)$, proportional to the turbulent magnetic field components

$$\frac{dX_i}{dt} = h_i(t) = v_z(t) \delta b_i(t) - v_i(t) \delta b_z(t), \quad (7)$$

2.2. The ensemble-averaged particle distribution function

The description of the cosmic ray transport within a large-scale guide magnetic field, which is uniform on the scales of the cosmic ray particles gyradii $R_L = v/|\Omega|$, is given by the solution of the Vlasov (collision-free Boltzmann) equation for the particle distribution function F (Hall and Sturrock 1968, Schlickeiser 2002). In spherical momentum coordinates (X, Y, z, p, μ, ϕ) the Vlasov equation reads (Hall and Sturrock 1968, Achatz et al. 1991)

$$\frac{\partial F}{\partial t} + v\mu \frac{\partial F}{\partial z} - \Omega \frac{\partial F}{\partial \phi} + p^{-2} \frac{\partial}{\partial p} [p^2 h_p(t) F] + \frac{\partial}{\partial y_\alpha} [h_\alpha(t) F] - Q_0(z, X, Y, p, \mu, \phi, t) = 0, \quad (8)$$

where $y_\alpha \in [X, Y, \mu, \phi]$ and

$$Q_0(z, X, Y, p, \mu, \phi, t) = S_0(z, X, Y, p, \mu, \phi, t) - \mathcal{N}_0 F - \mathcal{R}_0 F \quad (9)$$

accounts for sources and sinks (S_0) and the effects of the mirror force (\mathcal{N}_0) and momentum loss processes (\mathcal{R}_0), where the latter two operate on much longer spatial and time scales than the particle interactions with the stochastic fields. In Eq. (8) we use the Einstein sum convention for indices and the short notation $\partial_\nu = (\partial/\partial x_\nu)$. $x_{\nu,\sigma} \in [\mu, p, X, y]$ represent the four phase space variables μ, p, X, Y with non-vanishing stochastic fields $h_\nu(t)$.

The particle distribution function, $F(X, Y, z, p, \mu, \phi)$, varies in an irregular way under the influence of the stochastically fluctuating fields, $h_\nu(t)$. However, we do not look for the detailed function F , but rather we look for an ensemble-averaged solution, $\langle F \rangle$, an expectation value of of Equ. 8, which results from averaging over different realizations of the fields $h_\nu(t)$ with the same statistical averages. In the following treatment we will keep only first-order terms in the fluctuating quantities δF

$$F = \langle F \rangle + \delta F \quad (10)$$

and in the turbulent fields, $h_\nu(t)$. In other words we will consider the case of weak turbulence or quasilinear approximation. Moreover we will also neglect electric fields ($h_p = 0$). As shown in details in Paper 1, under the assumption of weak turbulence the ensemble-averaged solution, $\langle F \rangle$, can be obtained by solving the kinetic equation

$$\begin{aligned} \partial_t \langle F \rangle + v\mu \partial_z \langle F \rangle - \Omega \partial_\phi \langle F \rangle - Q_0(z, X, Y, p, \mu, \phi, t) = & - \frac{\partial}{\partial x_\alpha} P_{\alpha\sigma} \frac{\partial \langle F \rangle}{\partial x_\sigma} \\ & - \frac{\partial}{\partial \phi} P_{\phi\sigma} \frac{\partial \langle F \rangle}{\partial x_\sigma} - \frac{\partial}{\partial x_\alpha} P_{\alpha\sigma} \frac{\partial \langle F \rangle}{\partial \phi} \end{aligned} \quad (11)$$

where $x_\nu \in [X, Y, \mu]$ and the Fokker-Planck coefficients are given as

$$P_{\alpha\sigma} = \langle h_\alpha(t) \int_{t_0}^t ds h_\sigma(s) \rangle \quad (12)$$

The time-integration operator in Equ. 12 is performed over a generalization of the unperturbed gyrocenter orbit in the uniform magnetic field with deviations of the gyrophase given by

$$X_s = X, Y_s = Y, Z_s = Z + v\mu(s - t), p_s = p, \mu_s = \mu, \phi_s = \phi - \Omega(s - t) + \delta\phi(t - s), \quad (13)$$

that contains the additional arbitrary gyrophase variation $\delta\phi(t - s)$, with $\delta\phi = 0$ for $s = t$.

Fourier transforming the stochastic force in space, the time integral in Equ. 12 becomes

$$\begin{aligned} \int_{t_0}^t ds h_\sigma(s) &= \int d^3k \int_{t_0}^t du H_\sigma(\vec{k}, s) \\ &\times \exp \left[i\vec{k} \cdot \vec{X} + v\mu k_\parallel(s - t) + ik_\perp v \sqrt{1 - \mu^2} \int^s dw \cos(\psi - \phi + \Omega(w - t) - \delta\phi(t - w)) \right] \end{aligned} \quad (14)$$

where the particle position is given as

$$\vec{x}(s) = \begin{pmatrix} X + v\sqrt{1 - \mu^2} \int^s dw \cos(\phi - \Omega(w - t) + \delta\phi(t - w)) \\ Y + v\sqrt{1 - \mu^2} \int^s dw \sin(\phi - \Omega(w - t) + \delta\phi(t - w)) \\ Z + v\mu(s - t) \end{pmatrix} \quad (15)$$

and where we have introduced cylindrical coordinates for the wavenumber vector $\vec{k} = (k_\perp \cos \psi, k_\perp \sin \psi, k_\parallel)$ and the particle velocity.

As explained in details in Paper 1, in the *small Larmor radius approximation* (Chew et al. 1956, Kennel and Engelmann 1962) the distribution functions are independent of ϕ to lowest order and can then be expanded as

$$\langle F \rangle = F_0 + \frac{F_1}{\Omega} \quad (16)$$

and the Larmor-phase-averaged equation becomes

$$\partial_t F_0 + v\mu\partial_z F_0 - Q(z, X, Y, p, \mu, t) = -\frac{\partial}{\partial x_\alpha} D_{\alpha\sigma} \frac{\partial F_0}{\partial x_\sigma} - \quad (17)$$

with the gyro-averaged source term

$$Q(z, X, Y, p, \mu, t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi Q_0(z, X, Y, p, \mu, \phi, t), \quad (18)$$

and the gyro-averaged Fokker-Planck coefficients

$$D_{\alpha\sigma} = \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi P_{\alpha\sigma} = \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi < h_\alpha(t) \int_{t_0}^t ds h_\sigma^*(s) >, \quad (19)$$

where we replaced $h_\sigma(t) = h_\sigma^*(t)$ by its complex conjugate because the stochastic forces are real-valued quantities.

The generalization of the time integral in Eq. 12 from the unperturbed motion of the gyrocenters in the guide magnetic field to arbitrary gyrophase motions of particles is possible essentially because of the gyrophase-averaging in Eq. 19. As demonstrated in Paper 1 the considered general particle gyrophase motion then only modifies the arguments of trigonometric and Bessel functions as compared to the quasilinear approximation of particle orbits.

3. Derivation of Fokker-Planck coefficients for compressible magnetic turbulence

Following the approach of Paper 1 we now derive the Fokker-Planck coefficients for the case of compressible magnetic turbulence. We make the following assumptions on the nature of the turbulence: the turbulence is quasi-stationary, meaning that the correlation function

$\langle h_\nu^*(t)h_\sigma(s) \rangle$ depends only on the absolute value of the time difference $|t - s| = |\tau|$, so that with the substitution $s = t - \tau$ we find for Eq.(19)

$$D_{\nu\sigma} = \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{t_0}^t ds \langle h_\nu(t)h_\sigma^*(s) \rangle = \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{t-t_0} ds \langle h_\nu(t)h_\sigma^*(t-\tau) \rangle \quad (20)$$

As second assumption we use that the turbulent magnetic fields are homogenously distributed, meaning that independent from the actual position of the gyrocenter at time t the particles are subject to turbulence realizations with equal statistical properties. This allows us to average the Fokker-Planck coefficients over the spatial position of the guiding center using

$$\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3X e^{i(\vec{k}' - \vec{k}) \cdot \vec{X}} = \delta(\vec{k}' - \vec{k}), \quad (21)$$

implying that turbulence fields at different wavevectors are uncorrelated. As explained in details in Paper 1, the Fokker-Planck coefficients then become

$$\begin{aligned} D_{\nu\sigma} = & \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi \int d^3k \int_0^{t-t_0} d\tau \langle H_\nu(\vec{k}, t) H_\sigma^*(\vec{k}, t - \tau) e^{iv\mu k_\parallel \tau} \\ & \times \exp \left[-ik_\perp v \sqrt{1 - \mu^2} \left(\int_0^{t-\tau} dw \cos(\psi - \phi + \Omega(w - t) - \delta\phi(t - w)) + \frac{\sin(\phi - \psi)}{\Omega} \right) \right] \rangle \end{aligned} \quad (22)$$

A third assumption concerns the nature of the particle orbits, i.e. we consider only orbits where $\delta\phi(w)$ does not depend upon the fluctuating fields, so that the ensemble averaging in Eq. (22) involves only the 2nd order correlation functions of the stochastic fields. This is generally called the Corrsin independence hypothesis (Corrsin 1959, Salu and Montgomery, McComb 1990).

With $\xi = t - w$ and the abbreviation

$$G(\xi) = \Omega\xi + \delta\phi(\xi) \quad (23)$$

the Fokker-Planck coefficients (22) then are

$$D_{\nu\sigma} = \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi \int d^3k \int_0^{t-t_0} d\tau < H_\nu(\vec{k}, t) H_\sigma^*(\vec{k}, t - \tau) > \\ \times \exp \left[v\mu k_{\parallel} \tau + v k_{\perp} \sqrt{1 - \mu^2} \left(\int_0^{\tau} d\xi \cos(\phi - \psi + G(\xi)) - \frac{\sin(\phi - \psi)}{\Omega} \right) \right], \quad (24)$$

Later we will also assume that the turbulence has a finite decorrelation time t_c such that the correlation functions $< h_\nu(t) h_\sigma^*(t - \tau) > \rightarrow 0$ fall to a negligible magnitude for $\tau \rightarrow \infty$, so that the upper integration boundary in the τ -integral can be replaced by infinity

$$D_{\nu\sigma} = \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{\infty} d\tau < h_\nu(t) h_\sigma^*(t - \tau) >. \quad (25)$$

We remark that diffusive transport of cosmic rays happens if the turbulence is quasi-stationary and has a finite decorrelation time t_c , because the resulting gyro-averaged Fokker-Planck coefficients at large times $t - t_0 \gg t_c$ no longer depend on the time-difference $t - t_0$.

The equations of motion of the guiding center (Eqs. 7) can be written as

$$\frac{dX_i}{dt} = h_i(t) = v\mu\delta b_i(t) - v_i(t)\delta b_z(t), \quad (26)$$

if

$$v_i(t) = v\sqrt{1 - \mu^2} \cos\left((i - 1)\frac{\pi}{2} - \phi\right) \quad (27)$$

where $i = [1, 2]$, whereas the pitch-angle random force (Eq. (3)) is

$$\frac{d\mu}{dt} = h_\mu(t) = \Omega\sqrt{1 - \mu^2} (\cos\phi\delta b_2(t) - \sin\phi\delta b_1(t)), \quad (28)$$

3.1. Individual gyro-averaged Fokker-Planck coefficients

The Fourier transforms of the stochastic fields in (26) and (28) are

$$\begin{aligned}
H_i(\vec{k}, t) &= v\mu b_i(\vec{k}, t) - v_i(t)b_z(\vec{k}, t), \\
H_i^*(\vec{k}, t - \tau) &= v\mu b_i^*(\vec{k}, t - \tau) - v_i(t - \tau)b_z^*(\vec{k}, t - \tau), \\
H_\mu(\vec{k}, t) &= \Omega\sqrt{1 - \mu^2} \left(\cos(\phi)b_2(\vec{k}, t) - \sin(\phi)b_1(\vec{k}, t) \right) \\
H_\mu^*(\vec{k}, t - \tau) &= \Omega\sqrt{1 - \mu^2} \left(\cos(\phi + G(\tau))b_2^*(\vec{k}, t - \tau) - \sin(\phi + G(\tau))b_1^*(\vec{k}, t - \tau) \right) \quad (29)
\end{aligned}$$

where

$$v_i(t) = v\sqrt{1 - \mu^2} \cos\left((i - 1)\frac{\pi}{2} - \phi\right), \quad v_i(t - \tau) = v\sqrt{1 - \mu^2} \cos\left((i - 1)\frac{\pi}{2} - (\phi + G(\tau))\right)$$

In terms of the magnetic field correlation tensor

$$\langle b_i(\vec{k}, t)b_j^*(\vec{k}, t - \tau) \rangle = P_{ij}(\vec{k}, \tau) \quad (30)$$

we then obtain for the perpendicular Fokker-Planck coefficients

$$\begin{aligned}
D_{ij} &= \Re \frac{v^2}{2\pi} \int d^3k \int_0^{t-t_0} d\tau e^{iv\mu k_{\parallel}\tau} \int_0^{2\pi} d\phi H_{ij}(\vec{k}, \tau) \\
&\times \exp \left[ik_{\perp}v\sqrt{1 - \mu^2} \left(\int^\tau d\xi \cos(\phi - \psi + G(\xi)) - \frac{\sin(\phi - \psi)}{\Omega} \right) \right], \quad (31)
\end{aligned}$$

where

$$\begin{aligned}
H_{ij}(\vec{k}, \tau) &= \mu^2 P_{ij}(\vec{k}, \tau) - \mu\sqrt{1 - \mu^2} \cos\left((j - 1)\frac{\pi}{2} - (\phi + G(\tau))\right) P_{iz}(\vec{k}, \tau) + \\
&- \mu\sqrt{1 - \mu^2} \cos\left((i - 1)\frac{\pi}{2} - \phi\right) P_{zj}(\vec{k}, \tau) + \\
&+ (1 - \mu^2) \cos\left((i - 1)\frac{\pi}{2} - \phi\right) \cos\left((j - 1)\frac{\pi}{2} - (\phi + G(\tau))\right) P_{zz}(\vec{k}, \tau) \quad (32)
\end{aligned}$$

The mixed Fokker-Planck coefficients are instead given as

$$D_{i\mu} = \Re \frac{v\Omega\sqrt{1-\mu^2}}{2\pi} \int d^3k \int_0^{t-t_0} d\tau e^{iv\mu k_{\parallel}\tau} \int_0^{2\pi} d\phi H_{i\mu}(\vec{k}, \tau) \\ \times \exp \left[ik_{\perp} v \sqrt{1-\mu^2} \left(\int^{\tau} d\xi \cos(\phi - \psi + G(\xi)) - \frac{\sin(\phi - \psi)}{\Omega} \right) \right], \quad (33)$$

where

$$H_{i\mu}(\vec{k}, \tau) = \mu \cos(\phi + G(\tau)) P_{i2}(\vec{k}, \tau) - \mu \sin(\phi + G(\tau)) P_{i1}(\vec{k}, \tau) + \\ - \sqrt{1-\mu^2} \cos \left((i-1)\frac{\pi}{2} - \phi \right) \cos(\phi + G(\tau)) P_{z2}(\vec{k}, \tau) + \\ + \sqrt{1-\mu^2} \cos \left((i-1)\frac{\pi}{2} - \phi \right) \sin(\phi + G(\tau)) P_{z1}(\vec{k}, \tau)$$

and

$$D_{\mu i} = \Re \frac{v\Omega\sqrt{1-\mu^2}}{2\pi} \int d^3k \int_0^{t-t_0} d\tau e^{iv\mu k_{\parallel}\tau} \int_0^{2\pi} d\phi H_{\mu i}(\vec{k}, \tau) \\ \times \exp \left[ik_{\perp} v \sqrt{1-\mu^2} \left(\int^{\tau} d\xi \cos(\phi - \psi + G(\xi)) - \frac{\sin(\phi - \psi)}{\Omega} \right) \right], \quad (34)$$

where

$$H_{\mu i}(\vec{k}, \tau) = \mu \cos \phi P_{2i}(\vec{k}, \tau) - \mu \sin \phi P_{1i}(\vec{k}, \tau) + \\ - \sqrt{1-\mu^2} \cos \phi \cos \left((i-1)\frac{\pi}{2} - (\phi + G(\tau)) \right) P_{2z}(\vec{k}, \tau) + \\ + \sqrt{1-\mu^2} \sin \phi \cos \left((i-1)\frac{\pi}{2} - (\phi + G(\tau)) \right) P_{1z}(\vec{k}, \tau) \quad (35)$$

The Fokker-Planck coefficients parallel to the direction of the guide magnetic field are

$$D_{\mu\mu} = \Re \frac{\Omega^2(1-\mu^2)}{2\pi} \int d^3k \int_0^{t-t_0} d\tau e^{iv\mu k_{\parallel}\tau} \int_0^{2\pi} d\phi H_{\mu\mu}(\vec{k}, \tau) \\ \times \exp \left[ik_{\perp} v \sqrt{1-\mu^2} \left(\int^{\tau} d\xi \cos(\phi - \psi + G(\xi)) - \frac{\sin(\phi - \psi)}{\Omega} \right) \right], \quad (36)$$

where

$$H_{\mu\mu}(\vec{k}, \tau) = \cos \phi \cos(\phi + G(\tau)) P_{22}(\vec{k}, \tau) + \sin \phi \sin(\phi + G(\tau)) P_{11}(\vec{k}, \tau)$$

$$-\sin \phi \cos(\phi + G(\tau))P_{12}(\vec{k}, \tau) - \cos \phi \sin(\phi + G(\tau))P_{21}(\vec{k}, \tau))$$

The ϕ -integrals are calculated in Appendix A of Paper 1 and yield

$$D_{ij} = \Re \frac{v^2}{2} \int d^3k \int_0^{t-t_0} d\tau e^{v\mu k_{\parallel} \tau} I_{ij}(\vec{k}, \tau) \quad (37)$$

where

$$\begin{aligned} I_{ij}(\vec{k}, \tau) = & \left[2\mu^2 P_{ij}(\vec{k}, \tau) + (-1)^{j|i-j|} (1 - \mu^2) \cos \left(|i - j| \frac{\pi}{2} - G(\tau) \right) P_{zz}(\vec{k}, \tau) \right] J_0(Z) + \\ & - 2i\sqrt{1 - \mu^2} \mu [(-1)^{i-1} \sin \left((i-1) \frac{\pi}{2} - (\psi + \arcsin(\frac{Z_1}{Z})) \right) P_{zj}(\vec{k}, \tau) + \\ & (-1)^{j-1} \sin \left((j-1) \frac{\pi}{2} - (\psi + G(\tau) + \arcsin(\frac{Z_1}{Z})) \right) P_{iz}(\vec{k}, \tau)] J_1(Z) + \\ & (-1)^{(i-1)(j-1)} (1 - \mu^2) P_{zz}(\vec{k}, \tau) \cos \left(|i - j| \frac{\pi}{2} - (2\psi + G(\tau) + \arcsin(\frac{Z_1}{Z})) \right) J_2(Z) \end{aligned} \quad (38)$$

$$D_{i\mu} = \Re \frac{v\Omega\sqrt{1 - \mu^2}}{2} \int d^3k \int_0^{t-t_0} d\tau e^{v\mu k_{\parallel} \tau} I_{i\mu}(\vec{k}, \tau) \quad (39)$$

where

$$\begin{aligned} I_{i\mu}(\vec{k}, \tau) = & \sqrt{1 - \mu^2} \left[\sin \left((i-1) \frac{\pi}{2} - G(\tau) \right) P_{z1}(\vec{k}, \tau) + (-1)^i \cos \left((i-1) \frac{\pi}{2} - G(\tau) \right) P_{z2}(\vec{k}, \tau) \right] J_0(Z) \\ & + 2i\mu \left[\sin \left(\psi + G(\tau) + \arcsin(\frac{Z_1}{Z}) \right) P_{i2}(\vec{k}, \tau) + \cos \left(\psi + G(\tau) + \arcsin(\frac{Z_1}{Z}) \right) P_{i1}(\vec{k}, \tau) \right] J_1(Z) + \\ & \sqrt{1 - \mu^2} \left[(-1)^{i-1} \sin \left((i-1) \frac{\pi}{2} - (2\psi + G(\tau) + \arcsin(\frac{Z_1}{Z})) \right) P_{z1}(\vec{k}, \tau) + \right. \\ & \left. - \cos \left((i-1) \frac{\pi}{2} - (2\psi + G(\tau) + \arcsin(\frac{Z_1}{Z})) \right) P_{z2}(\vec{k}, \tau) \right] J_2(Z) \end{aligned} \quad (40)$$

$$D_{\mu i} = \Re \frac{v\mu\Omega\sqrt{1 - \mu^2}}{2} \int d^3k \int_0^{t-t_0} d\tau e^{v\mu k_{\parallel} \tau} I_{\mu i}(\vec{k}, \tau) \quad (41)$$

where

$$\begin{aligned}
I_{\mu i}(\vec{k}, \tau) = & \sqrt{1 - \mu^2} \left[(-1)^i \sin \left((i - 1) \frac{\pi}{2} - G(\tau) \right) P_{1z}(\vec{k}, \tau) - \cos \left((i - 1) \frac{\pi}{2} - G(\tau) \right) P_{2z}(\vec{k}, \tau) \right] J_0(Z) \\
& + 2i\mu \left[\sin \left(\psi + \arcsin\left(\frac{Z_1}{Z}\right) \right) P_{2i}(\vec{k}, \tau) + \cos \left(\psi + \arcsin\left(\frac{Z_1}{Z}\right) \right) P_{1i}(\vec{k}, \tau) \right] J_1(Z) + \\
& \sqrt{1 - \mu^2} \left[(-1)^{i-1} \sin \left((i - 1) \frac{\pi}{2} - (2\psi + G(\tau) + \arcsin\left(\frac{Z_1}{Z}\right)) \right) P_{1z}(\vec{k}, \tau) + \right. \\
& \left. - \cos \left((i - 1) \frac{\pi}{2} - (2\psi + G(\tau) + \arcsin\left(\frac{Z_1}{Z}\right)) \right) P_{2z}(\vec{k}, \tau) \right] J_2(Z)
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
D_{\mu\mu} = & \Re \frac{\Omega^2(1 - \mu^2)}{2} \int d^3k \int_0^{t-t_0} d\tau e^{iv\mu k_{\parallel} \tau} \\
& \times \left[\left(\cos(G(\tau)) J_0(Z) - \cos(2\psi + G(\tau) + 2 \arcsin\left(\frac{Z_1}{Z}\right)) J_2(Z) \right) P_{11}(\vec{k}, \tau) \right. \\
& + \left(\cos(G(\tau)) J_0(Z) + \cos(2\psi + G(\tau) + 2 \arcsin\left(\frac{Z_1}{Z}\right)) J_2(Z) \right) P_{22}(\vec{k}, \tau) \\
& - \left(\sin(G(\tau)) J_0(Z) + \sin(2\psi + G(\tau) + 2 \arcsin\left(\frac{Z_1}{Z}\right)) J_2(Z) \right) P_{21}(\vec{k}, \tau) \\
& \left. + \left(\sin(G(\tau)) J_0(Z) - \sin(2\psi + G(\tau) + 2 \arcsin\left(\frac{Z_1}{Z}\right)) J_2(Z) \right) P_{12}(\vec{k}, \tau) \right],
\end{aligned} \tag{43}$$

respectively, where $J_n(Z)$ denotes the Bessel function of the first kind and order n ,

$$Z_1 = k_{\perp} v \sqrt{1 - \mu^2} \int^{\tau} d\xi \cos(G(\xi)) \tag{44}$$

and

$$Z = k_{\perp} v \sqrt{1 - \mu^2} \left[\left(\int^{\tau} d\xi \cos(G(\xi)) \right)^2 + \left(\frac{1}{\Omega} + \int^{\tau} d\xi \sin(G(\xi)) \right)^2 \right]^{1/2} \tag{45}$$

4. Axisymmetric turbulence

Useful formulas can be obtained by assuming that the turbulence is asymmetric, meaning $P_{\alpha\beta}(\vec{k}, \tau)$ are independent of the wave phase ψ

$$P_{\alpha\beta}(\vec{k}, \tau) = P_{\alpha\beta}(k_{\parallel}, k_{\perp}, \tau), \quad (46)$$

The integration over ψ of the general formulas (38) - (43) then provides

$$D_{ij} = \Re \pi v^2 \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau e^{iv\mu k_{\parallel} \tau} J_0(Z) \\ \times \left[2\mu^2 P_{ij}(k_{\parallel}, k_{\perp}, \tau) + (-1)^{j|i-j|} (1 - \mu^2) \cos\left(\left|i - j\right| \frac{\pi}{2} - G(\tau)\right) P_{zz}(k_{\parallel}, k_{\perp}, \tau) \right], \quad (47)$$

$$D_{i\mu} = \Re \pi v \Omega (1 - \mu^2) \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau e^{iv\mu k_{\parallel} \tau} J_0(Z) \\ \times \left[\sin\left((i-1)\frac{\pi}{2} - G(\tau)\right) P_{z1}(k_{\parallel}, k_{\perp}, \tau) + (-1)^i \cos\left((i-1)\frac{\pi}{2} - G(\tau)\right) P_{z2}(k_{\parallel}, k_{\perp}, \tau) \right] \quad (48)$$

$$D_{\mu i} = \Re \pi v \Omega (1 - \mu^2) \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau e^{iv\mu k_{\parallel} \tau} J_0(Z) \\ \times \left[(-1)^i \sin\left((i-1)\frac{\pi}{2} - G(\tau)\right) P_{1z}(k_{\parallel}, k_{\perp}, \tau) - \cos\left((i-1)\frac{\pi}{2} - G(\tau)\right) P_{2z}(k_{\parallel}, k_{\perp}, \tau) \right] \quad (49)$$

and

$$D_{\mu\mu} = \Re \pi v^2 \Omega^2 (1 - \mu^2) \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau e^{iv\mu k_{\parallel} \tau} J_0(Z) \\ \times \left[\cos(G(\tau)) (P_{11}(k_{\parallel}, k_{\perp}, \tau) + P_{22}(k_{\parallel}, k_{\perp}, \tau)) + \sin(G(\tau)) (P_{12}(k_{\parallel}, k_{\perp}, \tau) - P_{21}(k_{\parallel}, k_{\perp}, \tau)) \right] \quad (50)$$

Introducing the left-handed and right-handed polarized stochastic magnetic field components

$$\delta b_{L,R} = \frac{1}{\sqrt{2}} [\delta b_1 \pm i \delta b_2], \quad (51)$$

so that

$$2P_{LL} = P_{11} + P_{22} + iP_{21} - iP_{12}, \quad 2P_{RR} = P_{11} + P_{22} + iP_{12} - iP_{21}, \quad (52)$$

we obtain for the pitch-angle Fokker-Planck coefficient (50)

$$\begin{aligned} D_{\mu\mu} = & \Re \pi \Omega^2 (1 - \mu^2) \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau J_0(Z) \\ & \times \left[e^{i(v\mu k_{\parallel} \tau + G(\tau))} P_{LL}(k_{\parallel}, k_{\perp}, \tau) + e^{i(v\mu k_{\parallel} \tau - G(\tau))} P_{RR}(k_{\parallel}, k_{\perp}, \tau) \right] \end{aligned} \quad (53)$$

5. Upper and lower limits of the general Fokker-Planck coefficients in the diffusion limit

If we now consider a magnetic field fluctuation decorrelation time $t_c = \gamma^{-1}$ (Schlickeiser and Achatz 1993, Bieber et al. 1994)

$$P_{ij}(\vec{k}, \tau) = P_{ij}^0(\vec{k}) e^{-\gamma\tau}, \quad (54)$$

then in the diffusion limit $t - t_0 \gg t_c$ the general Fokker-Planck coefficients (47) - (53) in asymmetric turbulence become

$$\begin{aligned} D_{\mu\mu} = & \pi \Omega^2 (1 - \mu^2) \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{\infty} d\tau J_0(Z) e^{-\gamma\tau} \\ & \times \left[\cos(v\mu k_{\parallel} \tau + G(\tau)) P_{LL}^0(k_{\parallel}, k_{\perp}) + \cos(v\mu k_{\parallel} \tau - G(\tau)) P_{RR}^0(k_{\parallel}, k_{\perp}) \right] \end{aligned} \quad (55)$$

and

$$D_{ij} = \pi v^2 \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{\infty} d\tau J_0(Z) e^{v\mu k_{\parallel} \tau - \gamma \tau} \\ \times \left[2\mu^2 P_{ij}^0(k_{\parallel}, k_{\perp}) + (1 - \mu^2) \frac{1}{2} (e^{\imath G(\tau)} + e^{-\imath G(\tau)}) P_{zz}^0(k_{\parallel}, k_{\perp}) \right] \quad (56)$$

$$D_{i\mu} = \pi v \Omega (1 - \mu^2) \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{\infty} d\tau J_0(Z) e^{v\mu k_{\parallel} \tau - \gamma \tau} \\ \times \left[\frac{1}{2} (e^{\imath G(\tau)} + e^{-\imath G(\tau)}) P_{z1}^0(k_{\parallel}, k_{\perp}) \delta_{i2} - \frac{1}{2} (e^{\imath G(\tau)} + e^{-\imath G(\tau)}) P_{z2}^0(k_{\parallel}, k_{\perp}) \delta_{i1} \right] \quad (57)$$

$$D_{\mu i} = \pi v \Omega \mu (1 - \mu^2) \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{\infty} d\tau J_0(Z) e^{v\mu k_{\parallel} \tau - \gamma \tau} \\ \times \left[\frac{1}{2} (e^{\imath G(\tau)} + e^{-\imath G(\tau)}) P_{1z}^0(k_{\parallel}, k_{\perp}) \delta_{i2} - \frac{1}{2} (e^{\imath G(\tau)} + e^{-\imath G(\tau)}) P_{2z}^0(k_{\parallel}, k_{\perp}) \delta_{i1} \right] \quad (58)$$

Note that in Eqs (55-58) we consider only the real part of the integral.

Because of the existence of the finite turbulence decorrelation time γ^{-1} , the correlation functions $P_{ij}(k_{\parallel}, k_{\perp}, \tau)$ fall to a negligible magnitude for $\tau \rightarrow \infty$, allowing us to replace the upper integration boundary in the τ -integrals in Eqs. (55), (56), (57) and (58) by infinity. We recover the diffusion limit which is valid for times $t - t_0 \gg \gamma^{-1}$.

If $J_0(A) \leq 1$ and $\cos(x) \leq 1$ Eqs. (55 - 58) become

$$D_{ij} < D_{ij}^{\max} = \frac{v^2 \mu^2}{\gamma} 2\pi \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} P_{ij}^0(k_{\parallel}, k_{\perp}) \int_0^{\infty} d\tau J_0(Z) \cos(v\mu k_{\parallel} \tau) e^{-\gamma \tau} \\ + \frac{v^2 (1 - \mu^2)}{2} 2\pi \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} P_{zz}^0(k_{\parallel}, k_{\perp}) \\ \int_0^{\infty} d\tau J_0(Z) \frac{e^{-\gamma \tau}}{2} \left[\cos(v\mu k_{\parallel} \tau + G(\tau)) + \cos(v\mu k_{\parallel} \tau - G(\tau)) \right] \delta_{ij}$$

$$= \frac{v^2 \mu^2 \delta b_{ij}^2}{\gamma} + \frac{v^2 (1 - \mu^2) \delta b_{zz}^2 \delta_{ij}}{2\gamma} \quad (59)$$

and

$$\begin{aligned} D_{i\mu} < D_{i\mu}^{\max} &= \frac{v\Omega(1 - \mu^2)}{2} 2\pi \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{\infty} d\tau J_0(Z) \frac{e^{-\gamma\tau}}{2} \\ &\quad \left[\left(\cos(v\mu k_{\parallel}\tau + G(\tau)) + \cos(v\mu k_{\parallel}\tau - G(\tau)) \right) \delta_{i2} P_{z1}^0(k_{\parallel}, k_{\perp}) \right. \\ &\quad \left. - \left[\cos(v\mu k_{\parallel}\tau + G(\tau)) + \cos(v\mu k_{\parallel}\tau - G(\tau)) \right] \delta_{i1} P_{z2}^0(k_{\parallel}, k_{\perp}) \right] \\ &= \frac{v\Omega(1 - \mu^2)}{2\gamma} [\delta_{i2} \delta b_{z1}^2 - \delta_{i1} \delta b_{z2}^2] \end{aligned} \quad (60)$$

$$\begin{aligned} D_{\mu i} < D_{i\mu}^{\max} &= \frac{v\Omega(1 - \mu^2)}{2} 2\pi \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{\infty} d\tau J_0(Z) \frac{e^{-\gamma\tau}}{2} \\ &\quad \left[\left[\cos(v\mu k_{\parallel}\tau + G(\tau)) + \cos(v\mu k_{\parallel}\tau - G(\tau)) \right] \delta_{i2} P_{1z}^0(k_{\parallel}, k_{\perp}) \right. \\ &\quad \left. - \left[\cos(v\mu k_{\parallel}\tau + G(\tau)) + \cos(v\mu k_{\parallel}\tau - G(\tau)) \right] \delta_{i1} P_{2z}^0(k_{\parallel}, k_{\perp}) \right] \\ &= \frac{v\Omega(1 - \mu^2)}{2\gamma} [\delta_{i2} \delta b_{1z}^2 - \delta_{i1} \delta b_{2z}^2] \end{aligned} \quad (61)$$

$$\begin{aligned} D_{\mu\mu} < D_{\mu\mu}^{\max} &= \frac{\Omega^2(1 - \mu^2)}{2\gamma} 2\pi \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} [P_{LL}^0(k_{\parallel}, k_{\perp}) + P_{RR}^0(k_{\parallel}, k_{\perp})] \\ &= \frac{\Omega^2(1 - \mu^2)}{2\gamma} [\delta b_{LL}^2 + \delta b_{RR}^2] = \frac{\Omega^2(1 - \mu^2)}{2\gamma} [\delta b_{xx}^2 + \delta b_{yy}^2], \end{aligned} \quad (62)$$

where

$$\delta b_{\mu\nu}^2 = \int d^3k P_{\mu\nu}^0(k_{\parallel}, k_{\perp}) = 2\pi \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} P_{\mu\nu}^0(k_{\parallel}, k_{\perp}) \quad (63)$$

According to the diffusion approximation (Schlickeiser 2002), neglecting the influence of the mirror force contribution \mathcal{N}_0 in Eq. (8), the perpendicular spatial diffusion coefficients

for the isotropic part of the cosmic ray phase space density are given by the pitch-angle average

$$\kappa_{ij} = \frac{1}{2} \int_{-1}^1 d\mu D_{ij}(\mu) \quad (64)$$

From Eq. 59 we find the upper limits

$$\kappa_{ij} < \kappa_{ij}^{\max} = \frac{v^2}{3\gamma} (\delta b_{ij}^2 + \delta b_{zz}^2 \delta_{ij}) \quad (65)$$

The parallel spatial diffusion coefficient for the isotropic part of the cosmic ray phase space density is given by the pitch-angle average

$$\kappa_{\parallel} = \frac{v^2}{8} \int_{-1}^1 d\mu \frac{(1 - \mu^2)^2}{D_{\mu\mu}(\mu)}, \quad (66)$$

and its lower limit does not change with respect to the incompressible case examined in Paper 1.

$$\kappa_{\parallel} > \kappa_{\parallel}^{\min} = \frac{\gamma v^2}{3\Omega^2 [\delta b_{xx}^2 + \delta b_{yy}^2]} \quad (67)$$

6. Mirror forces and turbulent scattering in the solar wind plasma

Mirror forces are produced by large-scale spatial variations of the guide magnetic field. The perpendicular component of the mirror force generates gradients and curvature drifts of the cosmic ray guiding center (Boyd and Sanderson 1969). In the presence of mirror forces the diffusion coefficients are given by the sum of the turbulent contribution, $k_{\mu\nu}^{(T)}$, and of the contribution due to the mirror forces, $k_{\mu\nu}^{(M)}$,

$$\kappa_{\mu\nu} = \kappa_{\mu\nu}^{(T)} + \kappa_{\mu\nu}^{(M)} \quad (68)$$

We will now compare the effect of mirror forces and of the turbulent contribution, $\kappa_{(\mu\nu)}^T$, calculated in Sect. 5, on the properties of cosmic ray transport in the solar wind plasma.

For mirror forces Schlickeiser and Jenko (2010) showed that in the case of a symmetric choice of the pitch-angle Fokker-Planck coefficients the ratio of the perpendicular mirror spatial diffusion coefficient to the parallel turbulent spatial diffusion coefficient is given by the derivatives of the cosmic-ray Larmor radius. In particular, considering the case of a magnetic power spectrum of Alfvénic slab turbulence $P(k) \propto k^{-s}$ with $s < 2$, Schlickeiser and Jenko 2010 obtained that the ratios of the non-zero perpendicular to parallel spatial diffusion coefficients are

$$\begin{aligned} \frac{\kappa_{XX}^{(M)}}{\kappa_{ZZ}^{(T)}} &= \frac{2-s}{6-s} \left(\frac{R_L}{3L_2} \right)^2 \\ \frac{\kappa_{YY}^{(M)}}{\kappa_{ZZ}^{(T)}} &= \frac{2-s}{6-s} \left(\frac{R_L}{3L_1} \right)^2 \end{aligned} \quad (69)$$

where we introduce the perpendicular magnetic field scale lengths (Schlickeiser and Jenko 2010)

$$\begin{aligned} L_1^{-1} &= -B^{-1} \frac{\partial B}{\partial x} \\ L_2^{-1} &= -B^{-1} \frac{\partial B}{\partial y} \end{aligned} \quad (70)$$

Summing the diagonal terms of the diffusion matrix we obtain

$$\frac{\kappa_{XX}^{(M)} + \kappa_{YY}^{(M)}}{\kappa_{ZZ}^{(T)}} = \frac{2-s}{6-s} \left(\frac{R_L}{3} \right)^2 \left(\frac{1}{L_1^2} + \frac{1}{L_2^2} \right) \quad (71)$$

where we remind that the cosmic ray gyroradius r_L is defined as

$$R_L = \frac{v}{\Omega} = \frac{pc}{ZeB_0} \quad (72)$$

From Eq. 71 we have

$$\kappa_{\parallel}^{\min} \left(\kappa_{XX}^{(M)} + \kappa_{YY}^{(M)} \right) > v^2 \left(\frac{R_L}{3} \right)^4 \frac{2-s}{6-s} \left(\frac{1}{L_1^2} + \frac{1}{L_2^2} \right) \quad (73)$$

since

$$\kappa_{\parallel}^{\min} = \frac{vR_L}{3}. \quad (74)$$

The relevant magnetic field random irregularities for the cosmic ray transport properties are the fast magnetosonic waves (Lee and Völk 1975, Cho and Lazarian, 2003). If we consider isotropic magnetosonic waves (Schlickeiser 2002)

$$\begin{aligned} P_{xx} &= \frac{g(k)}{8\pi k^2} \cos^2 \Theta \\ P_{yy} &= \frac{g(k)}{8\pi k^2} \\ P_{zz} &= \frac{g(k)}{8\pi k^2} \sin^2 \Theta \end{aligned} \quad (75)$$

then the random contributions to the field irregularities are

$$\begin{aligned} \delta b_{xx}^2 &= \frac{1}{4} \int_{-1}^1 d\mu \mu^2 \int dk g(k) = \frac{1}{6} \int dk g(k) \\ \delta b_{yy}^2 &= \frac{1}{4} \int_{-1}^1 d\mu \int dk g(k) = \frac{1}{2} \int dk g(k) \\ \delta b_{zz}^2 &= \frac{1}{4} \int_{-1}^1 d\mu (1 - \mu^2) \int dk g(k) = \frac{1}{3} \int dk g(k) \end{aligned} \quad (76)$$

Using the upper and lower limits for the diffusion coefficients from random turbulent forces in Eqs. (65) and (67) and applying it for the case of fast magnetosonic waves we have

$$\kappa_{\parallel}^{(T)\min} \left(\kappa_{xx}^{(T)\max} + \kappa_{xx}^{(T)\max} \right) \geq \left(\frac{vR_L}{3} \right)^2 \left[1 + \frac{2\delta b_{zz}^2}{\delta b_{xx}^2 + \delta b_{yy}^2} \right] = 2 \left(\frac{vR_L}{3} \right)^2. \quad (77)$$

Taking the ratio of Eq. 73 with Eq. 77 we obtain a relation for the product of perpendicular diffusion coefficients independent of κ_{ZZ}

$$\begin{aligned} \frac{\kappa_{XX}^{(M)} + \kappa_{YY}^{(M)}}{\kappa_{xx}^{(T)\max} + \kappa_{xx}^{(T)\max}} &> \left(\frac{R_L}{3} \right)^2 \frac{(L_1^2 + L_2^2)}{L_1^2 L_2^2} \frac{2-s}{2(6-s)} \\ &\sim \frac{2-s}{18(6-s)} \left(\frac{R_L}{\min[L_1, L_2]} \right)^2 \end{aligned} \quad (78)$$

Perpendicular spatial diffusion is thus dominated by turbulent forces at low particle momenta, where the gyroradius is less than the minimum of the perpendicular magnetic field focusing lengths. Alternatively, at high momenta, where the gyroradius is larger than the minimum of the perpendicular magnetic field focusing lengths, perpendicular diffusion is dominated by the mirror force contribution.

7. Summary and conclusions

In a large-scale magnetized plasma the description of cosmic ray transport is given by the solution of the Vlasov equation for the particle distribution function. The influence of stochastically fluctuating fields on the particle distribution function can be studied by looking for an ensemble-averaged solution of the Vlasov equation, which results from averaging over different realizations of turbulent fields with the same statistical properties.

In the small Larmor approximation it was shown in Paper 1 that one can obtain the solution of the Vlasov equation for arbitrary gyrophase motions of the particles, extending the quasilinear approximation to the particle orbit. In Paper 1 the transport parameters of energetic charged particles in turbulent magnetized cosmic plasmas were derived for the

case of an incompressible plasma, i.e. plasmas for which the component of the magnetic turbulence, $\delta B_z = 0$, parallel to the guide magnetic field, $\vec{B}_0 = B_0 \vec{e}_z$, is set to zero. Here we present the generalization of the theory to the case of compressible magnetic turbulence with $\delta B_z \neq 0$. Under the assumption that the turbulence is quasi-stationary and homogeneous we have obtained the gyro-averaged Fokker-Planck coefficients for a Corrsin type of generalized orbits. For an axisymmetric turbulence we have derived upper and lower limits for the perpendicular and pitch-angle Fokker-Planck coefficients in the diffusion limit. We have shown upper and lower limits for the perpendicular and parallel spatial diffusion coefficients, respectively, describing the spatial diffusion of the isotropic part of the cosmic ray phase space density. Finally using the upper and lower limits for the turbulent motion we compare the effects on the transport of cosmic ray particles of the turbulent and of mirror forces, if the latter cannot be neglected.

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8. Appendix: Quasilinear limit

Following the approach in Paper 1 and assuming $\delta\phi = 0$ for the particle orbit in Eq. 23

$$G(\xi) = \Omega\xi + \delta\phi(\xi) \quad (79)$$

we obtain the quasilinear approximation to the particle orbits (Shalchi and Schlickeiser 2004). The argument of the Bessel functions of first kind

$$Z = k_{\perp} v \sqrt{1 - \mu^2} \left[\left(\int^{\tau} d\xi \cos(G(\xi)) \right)^2 + \left(\frac{1}{\Omega} + \int^{\tau} d\xi \sin(G(\xi)) \right)^2 \right]^{1/2} \quad (80)$$

at order $n = 0$ becomes

$$Z = Z_0 = \frac{k_{\perp} v \sqrt{1 - \mu^2}}{\Omega} \left[\sin^2(\Omega\tau) + (1 - \cos^2(\Omega\tau))^2 \right]^{1/2} = \frac{k_{\perp} v \sqrt{1 - \mu^2}}{\Omega} [2(1 - \cos(\Omega\tau))]^{1/2} = \frac{2k_{\perp} v \sqrt{1 - \mu^2}}{\Omega} \left| \sin\left(\frac{\Omega\tau}{2}\right) \right| \quad (81)$$

The perpendicular Fokker-Planck coefficients (47) then become

$$D_{ij}^{QL} = \Re \pi v^2 \int_0^{\infty} d\tau \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} e^{i k_{\parallel} v_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin\left(\frac{\Omega\tau}{2}\right) \right| \right), \\ \left[2\mu^2 P_{ij}(k_{\parallel}, k_{\perp}, \tau) + (-1)^{|i-j|} (1 - \mu^2) \cos\left(\left| i - j \right| \frac{\pi}{2} - \Omega\tau\right) P_{zz}(k_{\parallel}, k_{\perp}, \tau) \right], \quad (82)$$

The mixed Fokker-Planck coefficients (48 and 49) are

$$D_{i\mu}^{QL} = \Re \pi v \Omega (1 - \mu^2) \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau e^{i v \mu k_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin\left(\frac{\Omega\tau}{2}\right) \right| \right) \\ \times \left[\sin\left((i-1)\frac{\pi}{2} - \Omega\tau\right) P_{z1}(k_{\parallel}, k_{\perp}, \tau) + (-1)^i \cos\left((i-1)\frac{\pi}{2} - \Omega\tau\right) P_{z2}(k_{\parallel}, k_{\perp}, \tau) \right] \quad (83)$$

$$D_{\mu i}^{QL} = \Re \pi v \Omega (1 - \mu^2) \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau e^{i v \mu k_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin\left(\frac{\Omega\tau}{2}\right) \right| \right) \\ \times \left[(-1)^i \sin\left((i-1)\frac{\pi}{2} - \Omega\tau\right) P_{1z}(k_{\parallel}, k_{\perp}, \tau) - \cos\left((i-1)\frac{\pi}{2} - \Omega\tau\right) P_{2z}(k_{\parallel}, k_{\perp}, \tau) \right] \quad (84)$$

whereas the Fokker-Planck coefficient (53) reduces to

$$D_{\mu\mu}^{QL} = \Re \pi \Omega^2 (1 - \mu^2) \int_0^{t-t_0} d\tau \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin\left(\frac{\Omega\tau}{2}\right) \right| \right) \\ \times \left[e^{i(v\mu k_{\parallel} + \Omega)\tau} P_{LL}(k_{\parallel}, k_{\perp}, \tau) + e^{i(v\mu k_{\parallel} - \Omega)\tau} P_{RR}(k_{\parallel}, k_{\perp}, \tau) \right] \quad (85)$$

Note that the quasilinear Fokker-Planck coefficients in axisymmetric turbulence no longer involve infinite sums of products of Bessel functions which enormously facilitates their numerical computation for specified turbulence field correlation tensors.

We now explicitly calculate the different Fokker-Planck coefficients

$$D_{xx}^{QL} = \Re \pi v^2 \int_0^\infty d\tau \int_{-\infty}^\infty dk_{\parallel} \int_0^\infty dk_{\perp} k_{\perp} e^{ik_{\parallel} v_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin \left(\frac{\Omega \tau}{2} \right) \right| \right),$$

$$\left[2\mu^2 P_{xx}(k_{\parallel}, k_{\perp}, \tau) + (1 - \mu^2) \cos(\Omega \tau) P_{zz}(k_{\parallel}, k_{\perp}, \tau) \right], \quad (86)$$

$$D_{xy}^{QL} = \Re \pi v^2 \int_0^\infty d\tau \int_{-\infty}^\infty dk_{\parallel} \int_0^\infty dk_{\perp} k_{\perp} e^{ik_{\parallel} v_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin \left(\frac{\Omega \tau}{2} \right) \right| \right),$$

$$\left[2\mu^2 P_{xy}(k_{\parallel}, k_{\perp}, \tau) + (1 - \mu^2) \sin(\Omega \tau) P_{zz}(k_{\parallel}, k_{\perp}, \tau) \right], \quad (87)$$

$$D_{yx}^{QL} = \Re \pi v^2 \int_0^\infty d\tau \int_{-\infty}^\infty dk_{\parallel} \int_0^\infty dk_{\perp} k_{\perp} e^{ik_{\parallel} v_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin \left(\frac{\Omega \tau}{2} \right) \right| \right),$$

$$\left[2\mu^2 P_{yx}(k_{\parallel}, k_{\perp}, \tau) - (1 - \mu^2) \sin(\Omega \tau) P_{zz}(k_{\parallel}, k_{\perp}, \tau) \right], \quad (88)$$

$$D_{yy}^{QL} = \Re \pi v^2 \int_0^\infty d\tau \int_{-\infty}^\infty dk_{\parallel} \int_0^\infty dk_{\perp} k_{\perp} e^{ik_{\parallel} v_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin \left(\frac{\Omega \tau}{2} \right) \right| \right),$$

$$\left[2\mu^2 P_{yy}(k_{\parallel}, k_{\perp}, \tau) + (1 - \mu^2) \cos(\Omega \tau) P_{zz}(k_{\parallel}, k_{\perp}, \tau) \right], \quad (89)$$

$$D_{x\mu}^{QL} = \Re \pi v \Omega (1 - \mu^2) \int_{-\infty}^\infty dk_{\parallel} \int_0^\infty dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau e^{v\mu k_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin \left(\frac{\Omega \tau}{2} \right) \right| \right)$$

$$\times \left[-\sin(\Omega \tau) P_{zx}(k_{\parallel}, k_{\perp}, \tau) - \cos(\Omega \tau) P_{zy}(k_{\parallel}, k_{\perp}, \tau) \right] \quad (90)$$

$$D_{y\mu}^{QL} = \Re \pi v \Omega (1 - \mu^2) \int_{-\infty}^\infty dk_{\parallel} \int_0^\infty dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau e^{v\mu k_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin \left(\frac{\Omega \tau}{2} \right) \right| \right)$$

$$\times \left[\cos(\Omega \tau) P_{zx}(k_{\parallel}, k_{\perp}, \tau) + \sin(\Omega \tau) P_{zy}(k_{\parallel}, k_{\perp}, \tau) \right] \quad (91)$$

$$D_{\mu x}^{QL} = \Re \pi v \Omega (1 - \mu^2) \int_{-\infty}^\infty dk_{\parallel} \int_0^\infty dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau e^{v\mu k_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin \left(\frac{\Omega \tau}{2} \right) \right| \right)$$

$$\times \left[\sin(\Omega\tau) P_{xz}(k_{\parallel}, k_{\perp}, \tau) - \cos(\Omega\tau) P_{yz}(k_{\parallel}, k_{\perp}, \tau) \right] \quad (92)$$

$$\begin{aligned} D_{\mu y}^{QL} = & \Re \pi v \Omega (1 - \mu^2) \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{t-t_0} d\tau e^{i v \mu k_{\parallel} \tau} J_0 \left(\frac{2k_{\perp} v_{\perp}}{\Omega} \left| \sin \left(\frac{\Omega\tau}{2} \right) \right| \right) \\ & \times \left[\cos(\Omega\tau) P_{xz}(k_{\parallel}, k_{\perp}, \tau) - \sin(\Omega\tau) P_{yz}(k_{\parallel}, k_{\perp}, \tau) \right] \end{aligned} \quad (93)$$

Using the Bessel function addition theorem (see Appendix in Paper 1) with $r_1 = r_2 = 1$, $\lambda = k_{\perp} v_{\perp} / \Omega$ and $\theta = \Omega\tau$ then

$$J_0(Z_0) = J_0 \left(\frac{k_{\perp} v_{\perp}}{\Omega} [2(1 - \cos \Omega\tau)]^{1/2} \right) = \sum_{n=-\infty}^{\infty} J_n^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) e^{i n \Omega \tau} \quad (94)$$

so that the perpendicular Fokker-Planck coefficients can be written as

$$\begin{aligned} D_{xx}^{QL} = & \Re \pi v^2 \int_0^{t-t_0} d\tau \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \\ & [2\mu^2 P_{xx}(k_{\parallel}, k_{\perp}, \tau) \sum_{n=-\infty}^{\infty} e^{i(k_{\parallel} v_{\parallel} - n\Omega)\tau} J_n^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) + \frac{1}{2}(1 - \mu^2) \\ & \sum_{n=-\infty}^{\infty} e^{i(k_{\parallel} v_{\parallel} - n\Omega)\tau} \left(J_{n-1}^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) + J_{n+1}^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) \right) P_{zz}(k_{\parallel}, k_{\perp}, \tau)], \end{aligned} \quad (95)$$

$$\begin{aligned} D_{xy}^{QL} = & \Re \pi v^2 \int_0^{t-t_0} d\tau \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \\ & [2\mu^2 P_{xy}(k_{\parallel}, k_{\perp}, \tau) \sum_{n=-\infty}^{\infty} e^{i(k_{\parallel} v_{\parallel} - n\Omega)\tau} J_n^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) + \frac{1}{2i}(1 - \mu^2) \\ & \sum_{n=-\infty}^{\infty} e^{i(k_{\parallel} v_{\parallel} - n\Omega)\tau} \left(J_{n-1}^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) - J_{n+1}^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) \right) P_{zz}(k_{\parallel}, k_{\perp}, \tau)], \end{aligned} \quad (96)$$

$$D_{yx}^{QL} = \Re \pi v^2 \int_0^{t-t_0} d\tau \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp}$$

$$\begin{aligned}
& [2\mu^2 P_{yx}(k_{\parallel}, k_{\perp}, \tau) \sum_{n=-\infty}^{\infty} e^{i(k_{\parallel} v_{\parallel} - n\Omega)\tau} J_n^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) - \frac{1}{2i}(1 - \mu^2) \\
& \sum_{n=-\infty}^{\infty} e^{i(k_{\parallel} v_{\parallel} - n\Omega)\tau} \left(J_{n-1}^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) - J_{n+1}^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) \right) P_{zz}(k_{\parallel}, k_{\perp}, \tau)], \quad (97)
\end{aligned}$$

$$\begin{aligned}
D_{yy}^{QL} &= \Re \pi v^2 \int_0^{t-t_0} d\tau \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \\
& [2\mu^2 P_{yy}(k_{\parallel}, k_{\perp}, \tau) \sum_{n=-\infty}^{\infty} e^{i(k_{\parallel} v_{\parallel} - n\Omega)\tau} J_n^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) + \frac{1}{2}(1 - \mu^2) \\
& \sum_{n=-\infty}^{\infty} e^{i(k_{\parallel} v_{\parallel} - n\Omega)\tau} \left(J_{n-1}^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) - J_{n+1}^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) \right) P_{zz}(k_{\parallel}, k_{\perp}, \tau)], \quad (98)
\end{aligned}$$

Using the addition theorem (94) the pitch angle Fokker-Planck coefficient (85) becomes

$$\begin{aligned}
D_{\mu\mu}^{QL} &= \Re \pi \Omega^2 (1 - \mu^2) \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_{\parallel} \int_0^{\infty} dk_{\perp} k_{\perp} \int_0^{\infty} d\tau e^{-i(n\Omega + k_{\parallel} v_{\parallel})\tau} \\
& \times \left[J_{n-1}^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) P_{LL}(k_{\parallel}, k_{\perp}, \tau) + J_{n+1}^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right) P_{RR}(k_{\parallel}, k_{\perp}, \tau) \right], \quad (99)
\end{aligned}$$

The pitch angle coefficient in (99) does not change with respect to the case of incompressible plasma treated in Paper 1. As remarked in Paper 1 the pitch angle coefficient agrees exactly with Eq. (12.2.5) of Schlickeiser (2002) for incompressible, axisymmetric turbulence.

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